§3.2. multiplicities and local rings

F = irr. plane curve, $P \in F$ find multiplicity of P on F via $\mathcal{O}_P(F)$.

$\forall G \in k[X,Y]$. $g := G \bmod (F) \in \Gamma(F) = k[X,Y]/(F)$.

Thm. F = irr. curve, $P \in F$.

$L = aX + bY + c$ through P not tangent to F at P then

(1) $m_P(F) = \dim_k \left(m_P(F)^n / m_P(F)^{n+1} \right)$ $\underline{n \gg 0}$

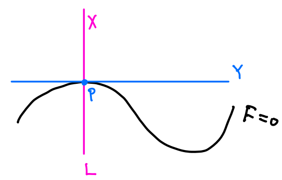
In particular, $m_P(F)$ depends only on $\mathcal{O}_P(F)$.

(2) P = simple $\Leftrightarrow$ $\mathcal{O}_P(F)$ = DVR

(3) if P simple then $\ell = L \bmod (F) \in \mathcal{O}_P(F)$ is a uniformizing parameter

Pf: affine change of coordinate

$\Rightarrow$ WMA: $\begin{cases} P = (0,0). \\ Y = \text{tangent line} \\ L = X \end{cases}$



$\underline{(2) \Rightarrow)}$ & $\underline{(3)}$

§2.5 Prop 4 $\Rightarrow$ ONTS: $m_P(F)$ is generated by x

$m_P(F) = (x, y)$ (P 2.43 & P 2.44)

$F = Y +$ higher terms $\Rightarrow$ $F = YG - X^2H$

G: 1 + higher terms

$\boxed{g(P) \neq 0 \Rightarrow g^{-1} \in \mathcal{O}_P(F)}$

$\Rightarrow$ $yg = x^2 h$ $\Rightarrow$ $y = x^2 h g^{-1} \in (x)$ $\Rightarrow$ $m_P(F) = (x)$.

$\underline{(2) \Leftarrow)}$: $\mathcal{O}_P(F) = \text{DVR} \xrightarrow{(1)} m_P(F) = 1$ $\Rightarrow$ (2) $\Leftarrow$)

(1): $0 \to m^n/m^{n+1} \to \mathcal{O}/m^{n+1} \to \mathcal{O}/m^n \to 0$

ONTS: $\exists s$ s.t. $\forall n \geq m_p(F)$

$$\dim_k(\mathcal{O}/m^n) = n \cdot m_p(F) + s$$

WMA: $P=(0,0)$, $m^n = I^n\mathcal{O}$ where $I=(x,y)$ (p 2.43)

$V(I^n) = \{P\} \Rightarrow k[X,Y]/(I^n,F) \cong \mathcal{O}_P(\mathbb{A}^2)/(I^n,F)\mathcal{O}_P(\mathbb{A}^2)$

$\cong \mathcal{O}/I^n\mathcal{O} \cong \mathcal{O}/m^n$

$\Rightarrow$ ONT: calculating $\dim k[X,Y]/(I^n,F)$

$m := m_p(F)$

$$0 \to k[X,Y]/I^{n-m} \xrightarrow{F} k[X,Y]/I^n \to k[X,Y]/(I^n,F) \to 0$$

$$\Rightarrow \dim_k(k[X,Y]/(I^n,F)) = nm - \frac{m(m-1)}{2}$$

F = (irr.) curve, P = simple pt on F.

ord_P^F := order function on $k(F)$ (or, simply ord_P)

$\mathrm{ord}_P^F(G) := \mathrm{ord}_P^F(G \bmod F)$

Fact: L = a line through P. P simple on F

$\mathrm{ord}_P^F(L) \geq 2 \Leftrightarrow L$ is tangent to F at P.

Pf: $\Rightarrow$) thm 1

$\Leftarrow$): reduce to the case in the proof of thm 1.

$L=Y$ tangent $\Rightarrow y = x^2hg^{-1} \Rightarrow \mathrm{ord}_P(y) \geq 2\,\mathrm{ord}_P(x) = 2$